ON THE RELIABILITY OF QUEUEING SYTEMS

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Introduction

Introduction

1. Main problem and motivation.
2. Network structure and main definitions.
3. Network reliability under conditions of heavy traffic.
4. Concluding remarks and future research.

Main problem and motivation

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1. *A queue management system is used to control queues. Queues of people form in various situations and locations in a queue area. The process of queue formation and propagation is defined as queueing theory.*

Network structure. Definition of arrival and service processes

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1. $G/GI/1$ computer network consisting from j stations, indexed by $j = 1, 2, \ldots, J$.
2. The basic components of the queueing network are arrival processes, service processes, and routing processes.
3. "First in, first out" (FIFO) service discipline is assumed for all J stations.
4. We assume that a station serves at its full capacity when the number of jobs waiting is equal to or exceeds the number of jobs theoretically capbable to process at the station.
5. $\{Z_n^{(j)},\ n \geq 1\}$ are J sequences of exogenous interarrival times, the random variable $Z_n^{(j)}$ is n-th interarrival time at station k,
6. Respectively,$\{S_n^{(j)},\ n \geq 1\}$ are sequences of service times, where $S_n^{(j)}$ is the n-th service time at station j.

Network structure

1. We define $\mu_j = (M[S_n^{(j)}])^{-1} > 0$ and $\lambda_j = (M[Z_n^{(j)}])^{-1} > 0, j = 1, 2, \ldots, k$;
2. Let $p_{ij} = P(\Phi_n^{(i)} = j) > 0;\ i, j = 1, 2, \ldots, k$.
3. This $k \times k$ matrix $P = (p_{ij})$ is called a *routing* matrix.
4. If $\Phi_n^{(i)} = j$ (which occurs with probability p_{ij}), then the n-th customer served at station i is routed to station j.
5. When $\Phi_n^{(i)} = 0$, the associated customer leaves the network.

Network structure of mixed-component Jackson queueing networks

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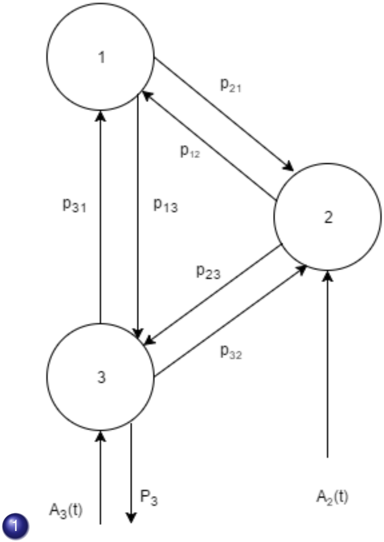
1. To construct renewal processes generated by the interarrival and service times, we introduce the summary lenght of interarrival times:
   $z_j(l) = \sum_{m=1}^{l} z_m^{(j)}, l \geq 1, j = 1, 2, \ldots, k.$
2. Summary lenght of service times is equal accordingly:
   $S_j(l) = \sum_{m=1}^{l} S_m^{(j)}, l \geq 1, j = 1, 2, \ldots, k.$
3. Let $A_j(t) = \max\left(l \geq 0 : z_j(l) \leq t\right)$ and $x_j(t) = \max\left(l \geq 0 : S_j(l) \leq t\right)$, (which indicates the room capacity of customers arrived and served at station j until time t)
4. Denote $\tau_j(t)$ as the total number of customers, who were departed from the jth station after service out of the network until time t.
5. We assume, that $P_j = 1 - \sum_{i=1}^{k} p_{ij}, p_j^t = 1 - \sum_{i=1}^{k} p_{ij}^t,$

Network structure of mixed-component Jackson queueing networks

1. At first, we divide the set of stations of the network into three sets:
   1) for $j = 1, 2, \ldots, m$ and $t > 0$, where $A_j(t) = 0$ and $P_j = 0$;
   2) for $j = m+1, m+2, \ldots, l$ and $t > 0$, where $A_j(t) > 0$ and $P_j = 0$;
   3) for $j = l+1, l+2, \ldots, k$ and $t > 0$, where $A_j(t) > 0$ and $P_j > 0$.
2. This is a definition of mixed-component Jackson queueing network.

Example of three-server model of mixed-component open queueing network



Network structure of mixed-component Jackson queueing networks

1. We assume, that $P_j = 1 - \sum_{i=1}^{k} p_{ij}, p_j^t = 1 - \sum_{i=1}^{k} p_{ij}^t,$
2. $\hat{y}_j(t) = A_j(t) - x_j(t) \cdot P_j$, (the part of the customers which has been routed to the other station or left the network entirely).
3. $k_j(t) = (k+1) \cdot \sup_{0 \leq s \leq t} (x_j(s) - \tau_j(t))$, (workload at each station)
4. To construct customer serving process at the j-th station of the mixed-component open queueing network, we calculate the workload capacity of each station: $\beta_j = 1 - \dfrac{\lambda_j + \sum_{i=1}^{k} \mu_i \cdot p_{ij}}{\mu_j}$ (less than zero means that the station is overloaded and the queue of customers is constantly growing)

Network behavior under conditions of heavy traffic

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1. We suppose that the following ("Heavy traffic") conditions are fulfilled:
$$\lambda_j + \sum_{i=1}^{k} \mu_i \cdot p_{ij} > \mu_j, \ \ j = 1, 2, \ldots, k. \tag{1}$$
2. The conditions guarantees that the queue and the virtual waiting time of a customer in the system is constantly growing.
3. One of the results of the paper is the probability limit theorem for the virtual waiting time $W_j(t)$ of a customer at the j-th station of the mixed-component open queueing network in time t.

Network behavior under conditions of heavy traffic

1. If conditions (1) are satisfied, then

$$\lim_{n\to\infty} P\left(\frac{W_j(nt)-\beta_j\cdot n\cdot t}{\hat{\sigma}_j\cdot\sqrt{n}} < x\right) = \int_{-\infty}^{x} \exp(-y^2/2)dy,$$

$0 \leq t \leq 1$ *and* $j = 1, 2, \ldots, k.$

(the virtual waiting time sequences of the customers form Wiener distribution).

2. Finally, if $t \geq \max\limits_{1\leq j\leq k} \dfrac{k_j}{\hat{\beta}_j}$ and conditions (1) are fulfilled, the computer network becomes unreliable (all the stations in the network fail).

Concluding remarks and future research

1. Conditions (1) mean that the summary length of jobs, arriving at the node of the network, is larger than the service of jobs at the same node of the network.
2. It is clear from this note that the length of jobs at the node of the network is constantly growing with probability one.
3. Conditions (1) are fundamental - the behaviour of the whole network and its evolution is not clear if they are not satisfied.

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